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Approximation by Polynomials with Restricted Ranges of their Derivatives*

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1. INTRODUCTION

Let X be a compact subset of a closed interval $[a, b]$ and assume that X contains at least $n + 1$ points for some fixed nonnegative integer n. Denote by $C(X)$ the space of all continuous real-valued functions defined on X. Let $\|f\| = \max_{x \in X} |f(x)|$ if $f \in C(X)$. Let $\{k_i\}_{i=1}^{p}$ be a fixed set of nonnegative integers satisfying $0 \leqslant k_1 < k_2 < \cdots < k_p \leqslant n$ and let $\{l_i\}_{i=1}^{p}$ and $\{u_i\}_{i=1}^{p}$ be fixed extended real-valued functions defined on X satisfying for each $i = 1,..., p$ the following conditions:

(i) l_i may take the value $-\infty$ but never $+\infty$.

(ii) u_i may take the value $+\infty$ but never $-\infty$.

(iii) $X_i^- = \{x \in X : l_i(x) = -\infty\}$ and $X_i^+ = \{x \in X : u_i(x) = +\infty\}$ are open subsets of X.

(iv) l_i continuous on $X - X_i^-$ and u_i is continuous on $X - X_i^+$.

(v) $l_i < u_i$ for all $x \in X$.

We note that, among other things, these assumptions assure the existence of an $\epsilon > 0$ for which $u_i - l_i \geqslant \epsilon$ for all $x \in X$ and all $i = 1,..., p$.

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Let Π_n be the collection of all algebraic polynomials of degree less than or equal to n, and define

$$K = \{\Phi \in \Pi_n : l_i(x) \leqslant \Phi^{(k_i)}(x) \leqslant u_i(x) \text{ for all } x \in X \text{ and } i = 1,..., p\}.$$

We shall always assume herein that K contains more than one function and also that there is a function $q_1 \in K$ satisfying $l_i(x) < q_1^{(k_i)}(x) < u_i(x)$ for all $x \in X$ and $i = 1,..., p$.

In this setting we will investigate the problem of approximating functions in $C(X)$ by functions in K. Thus for $f \in C(X)$ we shall say that $P \in K$ is a best approximation to f if $\| f - P \| \leqslant \| f - q \|$ for all $q \in K$. The existence of a best approximation corresponding to each $f \in C(X)$ follows from the fact that K is a closed subset of a compact subset of $C(X)$. The main problem studied in this paper is that of the characterization and uniqueness of these best approximations.

This paper is a generalization of the work of G. G. Lorentz and K. L. Zeller [2] and also of R. A. Lorentz [3] and of J. A. Roulier [4]. These papers study the problem when

$$l_i \equiv 0 \qquad \text{and} \qquad u_i \equiv +\infty$$

or

$$l_i \equiv -\infty \qquad \text{and} \qquad u_i \equiv 0$$

are the only possibilities.

It also generalizes the work of G. D. Taylor [5] and [6] in which $p = 1$ and $k_1 = 0$. The methods employed in this paper are essentially the same as those in [2] and [3] modified to fit our case.

2. Characterization of Best Approximations

We first introduce some special notation. Fix $f \in C(X)$ and $P \in K$. Let

$$E_+ = \{x \in X: f(x) - P(x) = \| f - p \|\},$$

$$E_- = \{x \in X: f(x) - P(x) = -\| f - p \|\},$$

$$E_+^{\,i} = \{x \in X: P^{(k_i)}(x) = l_i(x)\}, \qquad i = 1,..., p,$$

$$E_-^{\,i} = \{x \in X: P^{(k_i)}(x) = u_i(x)\}, \qquad i = 1,..., p.$$

These sets contain the "critical points" and will be used in our main characterization theorem. We always assume $f \notin K$. We note here and throughout that E_+, E_-, E_+^i, E_-^i all depend on f and P but this dependence will be suppressed in the notation unless absolutely necessary.

If $k_1 = 0$ then we may as well assume that $(E_+ \cup E_+^1) \cap (E_- \cup E_-^1) = \varnothing$, since otherwise it is easily seen that P is a best approximation for f from K. We note that in the most "natural" situation for $k_1 = 0$ [namely,

$$l_1(x) \leqslant f(x) \leqslant u_1(x)]$$

this is the case.

The proofs of the three characterization theorems which follow are omitted since they are essentially the same as the corresponding proofs in [2].

THEOREM 1. *Let $f \in C(X)$ and $P \in K$. Then P is a best approximation for f from K if and only if*

$$\max_{x \in E_+ \cup E_-} [f(x) - P(x)]\, q(x) \geqslant 0 \tag{1}$$

for each $q \in \Pi_n$ satisfying

$$l_i(x) \leqslant P^{(k_i)}(x) - q^{(k_i)}(x) \leqslant u_i(x) \tag{2}$$

for all $x \in X$ and $i = 1,..., p$.
[If $k_1 = 0$ we assume $(E_+ \cup E_+^1) \cap (E_- \cup E_-^1) = \varnothing$]

Our goal now is to alter this theorem to make it more useful in recognizing polynomials of best approximation. Our end result will be characterization theorems like those in [2] in terms of the nonexistence of solutions to certain Birkhoff interpolation problems. This, together with the interpolation theory of Atkinson and Sharma [1], will be the tool used in handling the problem of uniqueness.

THEOREM 2. *Let $f \in C(X)$ and $P \in K$. Then P is a best approximation to f from K if and only if there is no polynomial $q \in \Pi_n$ satisfying*

$$(\operatorname{sgn}[f(x) - P(x)])\, q(x) < 0, \qquad \text{for} \quad x \in E_+ \cup E_- \tag{3}$$

and

$$\begin{aligned} q^{(k_i)}(x) &> 0 \qquad \text{on} \quad E_-^i, \\ q^{(k_i)}(x) &< 0 \qquad \text{on} \quad E_+^i, \quad i = 1,..., p. \end{aligned} \tag{4}$$

It is clear that we may replace (4) by

$$\begin{aligned} q^{(k_i)}(x) &\geqslant 0 \quad \text{on} \quad E_-^i, \\ q^{(k_i)}(x) &\leqslant 0 \quad \text{on} \quad E_+^i. \end{aligned} \tag{5}$$

We wish to improve this characterization once again. For brevity of notation in the following theorem we let

$$\sigma(x) = \operatorname{sgn}[f(x) - P(x)].$$

THEOREM 3. *A polynomial $P \in K$ is a polynomial of best approximation for a given $f \in C(X)$ if and only if there exist points $x_j \in E_+ \cup E_-$ $j = 1,..., u$; $y_{ij}^+ \in E_+^i$, $j = 1,..., \lambda_i^+$; $y_{ij}^- \in E_-^i$, $j = 1,..., \lambda_i^-$, $i = 1,..., p$ with*

$$u + \lambda_1^+ + \cdots + \lambda_p^+ + \lambda_1^- + \cdots + \lambda_p^- \leqslant n + 2 \tag{6}$$

for which there is no $q \in \Pi_n$ that satisfies

$$\sigma(x_j)\, q(x_j) < 0, \qquad j = 1,..., u, \tag{7}$$

$$q^{(k_i)}(y_{ij}^+) < 0, \qquad j = 1,..., \lambda_i^+, \quad i = 1,..., p, \tag{8}$$

$$q^{(k_i)}(y_{ij}^-) > 0, \qquad j = 1,..., \lambda_i^-, \quad i = 1,..., p, \tag{9}$$

or, equivalently, if and only if there exists such points x_j, y_{ij}^+, y_{ij}^- and corresponding constants $b_j > 0$, $b_{ij}^+ > 0$, $b_{ij}^- < 0$ for which

$$\sum_{j=1}^{u} b_j \sigma(x_j)\, q(x_j) + \sum_{i=1}^{p} \left\{ \sum_{j=1}^{\lambda_i^+} b_{ij}^+ q^{(k_i)}(y_{ij}^+) + \sum_{j=1}^{\lambda_i^-} b_{ij}^- q^{(k_i)}(y_{ij}^-) \right\} = 0 \tag{10}$$

holds for all polynomials $q \in \Pi_n$.

The proof of this theorem is the same as the proof of Theorem 3 in [2]. One makes use of a theorem of Caratheodory on convex hulls.

Note that in Theorem 3 we must have

$$u + (k_1 + 1)(\lambda_1^- + \lambda_1^+) + \cdots + (k_p + 1)(\lambda_p^- + \lambda_p^+) \geqslant n + 2.$$

Otherwise the Hermite interpolation problem is solvable, which assigns arbitrary values to q at the points x_j and to $q, q', ..., q^{(k_i)}$ at the points y_{ij}^+, y_{ij}^-.

Fix K corresponding to $0 \leqslant k_1 < k_2 < \cdots < k_p \leqslant n$ and $\{l_i\}_{i=1}^p$ and $\{u_i\}_{i=1}^p$ as above. Fix $f \in C(X)$. If $k_1 = 0$ then we shall assume that $l_1(x) \leqslant f(x) \leqslant u_1(x)$ in what follows. Then the set of all best approximations from K

to f is a compact, convex set $\mathscr{B}$ in $C(X)$. Among all polynomials in $\mathscr{B}$ we single out those with the smallest sets $E_+ \cup E_-$, $E_+{}^i$ and $E_-{}^i$.

DEFINITION. We call a polynomial $P_0 \in \mathscr{B}$ *minimal* for f if for any other $P \in \mathscr{B}$ we have degree of $P \leqslant$ degree of P_0,

$$E_+(P_0) \cup E_-(P_0) \subset E_+(P) \cup E_-(P),$$

$$E_+{}^i(P_0) \subset E_+{}^i(P), \qquad i = 1,..., p,$$

$$E_-{}^i(P_0) \subset E_-{}^i(P), \qquad i = 1,..., p;$$

and if, moreover, $P(x)$ and $P_0(x)$ coincide on $E_+(P_0) \cup E_-(P_0)$.

THEOREM 4. *For each $f \in C(X)$ there exists a minimal polynomial of best approximation from K. [As above, if $k_1 = 0$ we assume that $l_1(x) \leqslant f(x) \leqslant u_1(x)$.]*

Proof. Set $E_+ = \bigcap_{P \in \mathscr{B}} E_+(P)$ and $E_- = \bigcap_{P \in \mathscr{B}} E_-(P)$ for a fixed $f \in C(X)$. Also set $E_+{}^i = \bigcap_{P \in \mathscr{B}} E_+{}^i(P)$ and $E_-{}^i = \bigcap_{P \in \mathscr{B}} E_-{}^i(P)$, $i = 1,..., p$. If $\mathscr{B}$ consists of only one function then the theorem is trivially true. Thus assume $\mathscr{B}$ contains more than one polynomial. Fix i and consider $E_+{}^i$. If $P_1, P_2 \in \mathscr{B}$ then $t \in E_+{}^i$ implies $P_1^{(k_i)}(t) = P_2^{(k_i)}(t)$. Thus either $E_+{}^i$ is finite or $E_+{}^i = E_+{}^i(P)$ for any $P \in \mathscr{B}$. Similarly, $E_-{}^i$ is finite or $E_-{}^i = E_-{}^i(P)$ for any $P \in \mathscr{B}$. Thus we can find a finite number of polynomials $P_1,..., P_N \in \mathscr{B}$ for which

$$E_+{}^i = \bigcap_{j=1}^{N} E_+{}^i(P_j)$$

and

$$E_-{}^i = \bigcap_{j=1}^{N} E_-{}^i(P_j), \qquad i = 1,..., p,$$

Noting that E_+ and E_- are disjoint sets, we can show as above that both E_+ and E_- are finite sets. Thus there is a finite set of polynomials $Q_1,..., Q_M \in \mathscr{B}$ so that

$$E_+ = \bigcap_{j=1}^{M} E_+(Q_j) \qquad \text{and} \qquad E_- = \bigcap_{j=1}^{M} E_-(Q_j).$$

Thus taking the polynomials $P_1,..., P_N$ and $Q_1,..., Q_M$ and renumbering them as $R_1, \ldots, R_L$ we have

$$E_+ = \bigcap_{\nu=1}^{L} E_+(R_\nu),$$

$$E_- = \bigcap_{\nu=1}^{L} E_-(R_\nu),$$

$$E_+^i = \bigcap_{\nu=1}^{L} E_+^i(R_\nu), \qquad i = 1,\ldots, p,$$

$$E_-^i = \bigcap_{\nu=1}^{L} E_-^i(R_\cdot), \qquad i = 1,\ldots, p.$$

Now let $P^* = (1/L) \sum_{\nu=1}^{L} R_\nu$. Then $P^* \in \mathscr{B}$ and $E_+(P^*) = E_+$, $E_-(P^*) = E_-$, $E_+^i(P^*) = E_+^i$, $E_-^i(P^*) = E_-^i$, $i = 1,\ldots, p$. If degree of $P^* \geqslant$ degree of P for any other $P \in \mathscr{B}$, let $P^* = P_0$. Otherwise, let P_1 be an element in $\mathscr{B}$ of highest degree. Then $\frac{1}{2}(P^* + P_1) = P_0 \in \mathscr{B}$, degree $P_0 >$ degree P^*, and

$$E_+(P_0) = E_+, \qquad E_-(P_0) = E_-,$$

$$E_+^i(P_0) = E_+^i, \qquad E_-^i(P_0) = E_-^i, \qquad i = 1,\ldots, p.$$

Moreover, if P is any other element of $\mathscr{B}$ then P, P^*, and P_0 coincide on $E_+ \cup E_-$ and degree $P_0 \geqslant$ degree P. This completes the proof.

3. Uniqueness

Uniqueness in general does not hold for this problem. For example, if the u_i and l_i are not differentiable functions then we need not have a unique $P \in K$ of best approximation for a given $f \in C(X)$.

Let $X = [-1, 1]$ and $n = 2$. Assume

$$p = 1, \qquad k_p = k_1 = 1,$$

$$u_1(x) = 2 \qquad \text{and} \qquad l_1(x) = \begin{cases} x + 1 & \text{on} \quad [-1, 0] \\ -x + 1 & \text{on} \quad [0, 1]. \end{cases}$$

If $f(x) = -x$ then there is no unique best approximation for f from K for this problem. In fact, if $P_a(x) = ax^2 + x - a$ then for each $a \in [-\frac{1}{2}, +\frac{1}{2}]$ P_a is a best approximation to this f from K. We omit the proof of this statement since it is easily verified.

It is also easy to see that if $k_1 = 0$ and if $f(x) \leqslant l_1(x)$ or $f(x) \geqslant u_1(x)$ then unique best approximation need not occur in general.

So, to our assumptions (i)–(v) and the others in Section 1 we add the following assumptions:

(vi) $X = [a, b]$.

(vii) Either $u_i(x) = +\infty$ for all $x \in X$ or u_i is differentiable at each $x \in (a, b)$. Either $l_i(x) = -\infty$ for all $x \in X$ or l_i is differentiable at each $x \in (a, b)$.

(viii) In the case that $k_{i+1} = k_i + 1$ we have $u_i = +\infty$ or $u_i' = u_{i+1}$ or $u_i' = l_{i+1}$. Also in this case we have $l_i = -\infty$, $l_i' = u_{i+1}$ or $l_i' = l_{i+1}$.

(ix) If $k_1 = 0$ we assume $l_1(x) \leqslant f(x) \leqslant u_1(x)$.

We also have need of some additional notation:

l_+^i	is the number of elements in E_+^i.
l_-^i	is the number of elements in E_-^i.
m_+	is the number of elements in E_+.
m_-	is the number of elements in E_-.
e_+^i	is the number of elements in $E_+^i \cap \{a, b\}$.
e_-^i	is the number of elements in $E_-^i \cap \{a, b\}$.

Here, as before, we have suppressed the fact that E_+, E_-, E_+^i, E_-^i depend on f and P. Also, we allow the possibility of some of the above numbers being infinite.

As in [2] and in [3] the critical tool in studying uniqueness of best approximation is the notion of "free" or "poised" matrices and the corresponding Birkhoff interpolation problem, which we shall henceforth abbreviate as BIP. We will be as brief as possible in describing these problems, giving only the necessary notions and results pertinent to our situation. Let $E = (e_{ij})$ be an $m \times (n + 1)$ matrix $i = 1,..., m$; $j = 0,..., n$. We assume E has only ones and zeros as entries. Let $e = \{(i, j) \mid e_{ij} = 1\}$. The matrix E is called an *incidence matrix*. Even though it is usually assumed that E has exactly $(n + 1)$ nonzero entries we will dispense with this restriction for convenience, adding it in as a hypothesis where necessary.

If the number of nonzero entries is $n + 1$, then for any choice of real numbers $x_1 < x_2 < \cdots < x_m$ and b_{ij} for $(i, j) \in e$, we associate with E the following BIP, where Q is assumed to be a polynomial of degree n or less:

$$Q^{(j)}(x_i) = b_{ij} \qquad (i, j) \in e.$$

Similarly, if

$$Q^{(\mathscr{B}_{ij})}(y_j) = \delta_{ij} \qquad 0 \leqslant \mathscr{B}_{ij} \leqslant n$$

is a BIP for a polynomial $Q \in \Pi_n$ (with $n + 1$ conditions) then we may associate with this BIP an incidence matrix E with $(n + 1)$ nonzero entries. Let $\lambda_1 < \cdots < \lambda_m$ be the points y_j arranged in increasing order. We define $E = (e_{ij})$ where $e_{ij} = 1$ if $Q^{(j)}(\lambda_i)$ is one of the conditions and $e_{ij} = 0$ otherwise.

If such a BIP has a unique solution regardless of the choice of the x_i and the b_{ij}, then the associated incidence matrix E is said to be *free* or *poised*.

Let E be an incidence matrix and define $m_j = \sum_{i=1}^{m} e_{ij}$, $j = 0, 1,..., n$. Then E is said to satisfy the *Polya condition* if, for each $k = 0, 1,..., n$,

$$\sum_{j=0}^{k} m_j \geqslant k + 1. \qquad (*)$$

A *maximal sequence* of the incidence matrix E is a sequence of 1's $(e_{ij} ,..., e_{i,j+r})$ which can not be extended to a longer sequence of 1's in row i of E. This maximal sequence is a *supported maximal sequence* if there exist integers $0 \leqslant j_0, j_1 < j$ and $1 \leqslant i_0 < i < i_1 \leqslant m$ for which $e_{i_0 j_0} = e_{i_1 j_1} = 1$. If each supported maximal sequence has an even number of elements then E is said to satisfy the *Atkinson–Sharma* (A–S) *condition*. K. Atkinson and A. Sharma in [1] proved:

THEOREM 5. *If the* $m \times (n + 1)$ *incidence matrix* E [*with* $(n + 1)$ *nonzero entries*] *satisfies both the* A–S *and the Polya conditions then* E *is free.*

It is this theorem which will be used to study uniqueness of best approximation. It is used in much the same way as in [3].

In the next two lemmas we assume that $f \in C(X)$ and that P_0 is a fixed minimal polynomial of best approximation to f as described above. In addition E_+, E_-, E_+^i, E_-^i are the sets corresponding to this P_0 and this f.

LEMMA 1. *Let* $P \in \mathscr{B}$ *and define* $D = P_0 - P$. *Let* $\nu =$ *exact degree of* D. *Then*

$$D^{(k_j+1)}(y) = 0, \qquad y \in (E_+^j \cup E_-^j) \cap (a, b) \qquad j = 1,..., p. \qquad (11)$$

If $k_{j+1} = k_j + 1$ *for some* j, *where* $k_j \leqslant \nu$, *then*

$$E_+^j \cup E_-^j \subset \{a, b\}. \qquad (12)$$

Proof. Let $y \in E_+^j \cap (a, b)$, then

$$P_0^{(k_j)}(y) = l_j(y).$$

and

$$P^{(k_j)}(y) = l_j(y)$$

Moreover,

$$P_0^{(k_j+1)}(y) = l_j'(y)$$

otherwise $P_0^{(k_j)}(y) - l_j(y)$ would change sign at y. Similarly,

$$P^{(k_j+1)}(y) = l_j'(y),$$

Hence,

$$D^{(k_j+1)}(y) = 0.$$

We proceed similarly for $y \in E_-^j \cap (a, b)$.

Suppose $k_{j+1} = k_j + 1$ for some j with $k_j \leqslant \nu$. If $l_j \equiv -\infty$ then $E_+^j = \varnothing$. So, assume $l_j \neq -\infty$. It follows then, from assumptions (vii) and (viii) that both $P_0^{(k_j)} - l_j$ and $P^{(k_j)} - l_j$ are both increasing on $[a, b]$ or both decreasing on $[a, b]$. Without loss of generality assume that they are both increasing on $[a, b]$. If $y_0 \in (a, b) \cap E_+^j$ then $P_0^{(k_j)}(y_0) - l_j(y_0) = 0$ and so

$$P_0^{(k_j)}(y) - l_j(y) = 0 \qquad \text{for} \quad a < y \leqslant y_0 .$$

Thus $(a, y_0] \subset E_+^j$. Hence $P^{(k_j)}(y) - l_j(y) = 0$ for $a < y \leqslant y_0$. Thus we have $P_0^{(k_j)}(y) - P^{(k_j)}(y) = 0$ for $a < y \leqslant y_0$ and so $D^{(k_j)} \equiv 0$. But this is impossible since $k_j \leqslant \nu$. Hence $E_+^j \subset \{a, b\}$. Similarly we show $E_-^j \subset \{a, b\}$. This completes the proof of Lemma 1.

If $P \in \mathscr{B}$ and $P \neq P_0$ and $f \in C[a, b]$ we associate with P and f a certain incidence matrix E. We see that $E_+ \cup E_-$ is finite and we let ν represent the exact degree of $P_0 - P$. Moreover, let x_i, y_{ji}^+, y_{ji}^- represent the elements of $E_+ \cup E_-$, E_+^i and E_-^i, respectively, for $k_j \leqslant \nu$. This is possible since l_+^j and l_-^j are finite for $k_j \leqslant \nu$.

We now define the incidence matrix E corresponding to the following BIP:

(a) $Q(x_i) = \alpha_i \qquad i = 1,..., m_+ + m_-$;

(b) $Q^{(k_j)}(y_{ij}^+) = \beta_{ji} \qquad k_j \leqslant \nu,\ i = 1,..., l_+^j$;

(c) $Q^{(k_j)}(y_{ji}^-) = \gamma_{ji} \qquad k_j \leqslant \nu,\ i = 1,..., l_-^j$;

(d) $Q^{(k_j+1)}(y_{ij}^+) = \delta_{ji} \qquad a < y_{ji}^+ < b,\ k_j + 1 \leqslant \nu,\ i = 1,..., l_+^j$;

(e) $Q^{(k_j+1)}(y_{ji}^-) = \epsilon_{ji} \qquad a < y_{ji}^- < b,\ k_j + 1 \leqslant \nu,\ i = 1,..., l_-^j$.

In the case $k_1 = 0$ we fit conditions (b) and (c) to agree with (a) where necessary.

LEMMA 2. *The matrix E corresponding to conditions* (a)–(e) *satisfies the* A–S *condition and the Polya condition.*

Proof. We first note that conditions (b)–(e) do not overlap if $k_j \geqslant 1$. This follows easily from (11) and (12). Hence, for $k_j \geqslant 1$ and $a < y_{ji}^{+}$, $y_{ji}^{-} < b$ conditions (b)–(e) come in nonoverlapping pairs. If $k_1 = 0$ then overlapping is possible in the first column of E between condition (a) and conditions (b) and (c). But in these cases the 1 is not the beginning of a supported sequence. Thus E satisfies the A–S condition.

We will now show that E satisfies the Polya condition. Since $\|f - P_0\|$ is attained for at least one point $m_+ + m_- \geqslant 1$. Hence (*) is satisfied for $k = 0$. Assume (*) is not satisfied for some k, $0 < k \leqslant \nu$. Let k_0 be the smallest k for which (*) fails. Consider the incidence matrix E_0 that consists of the columns of E numbered from 0 to $(k_0 - 1)$. By assumption then

$$\sum_{j=0}^{k_0} m_j \leqslant k_0$$

and (*) is satisfied for $0 \leqslant k \leqslant k_0 - 1$. Thus

$$\sum_{j=0}^{k_0-1} m_j \geqslant k_0 .$$

Hence we have

$$\sum_{j=0}^{k_0-1} m_j = k_0 \qquad \text{and} \qquad m_{k_0} = 0.$$

Since the k_0-th column of E has only zeros, no maximal sequence of E can cross this column. Hence, E_0 must satisfy the A–S condition. Consider the BIP for a polynomial Q of degree $\leqslant k_0 - 1$ corresponding to E_0 with values

$$\begin{aligned}
Q(x_i) &= -\sigma(x_i) && i = 1,..., m_+ + m_-, \\
Q^{(k_j)}(y_{ji}^{+}) &= 0 && k_j \leqslant k_0 - 1,\ i = 1,..., l_+^{\,j}, \\
Q^{(k_j)}(y_{ji}^{-}) &= 0 && k_j \leqslant k_0 - 1,\ i = 1,..., l_-^{\,j}, \\
Q^{(k_j+1)}(y_{ji}^{+}) &= 0 && k_j + 1 \leqslant k_0 - 1,\ a < y_{ji}^{+} < b,\ i = 1,..., l_+^{\,j}, \\
Q^{(k_j+1)}(y_{ji}^{-}) &= 0 && k_j + 1 \leqslant k_0 - 1,\ a < y_{ji}^{-} < b,\ i = 1,..., l_-^{\,j}.
\end{aligned} \tag{13}$$

Remember that $\sigma(x) = \operatorname{sgn}[f(x) - P(x)]$.

We note that if $k_1 = 0$ and if $y_{1i}^{+} \in E_{+}$ then we define

$$Q^{(k_1)}(y_{1i}^{+}) = -1 = Q(x_l), \qquad x_l \in E_{+}.$$

Likewise if $y_{1i}^{-} \in E_{-}$ we define $Q^{(k_1)}(y_{1i}^{-}) = 1 = Q(x_l)$, $x_l \in E_{-}$. The cases $y_{1i}^{+} \in E_{-}$ and $y_{1i}^{-} \in E_{+}$ may not occur because of assumption (ix). That is $E_{-} \cap E_{+}^{1} = \varnothing$ and $E_{+} \cap E_{-}^{1} = \varnothing$ if $k_1 = 0$.

Thus no contradictions occur in (13) even if overlapping occurs in the 1st column. Since E_0 satisfies the Polya condition and the A–S condition it is poised. Hence a unique polynomial Q of degree $\leqslant k_0 - 1$ satisfying (13) exists. But if $k > k_0 - 1$, $Q^{(k)}(x) = 0$. Hence (3) and (5) are violated; a contradiction. And so, E satisfies the Polya condition.

THEOREM 6. *Let $f \in C[a, b]$, $n \geqslant 0$ and K be as above with the additional restrictions* (vi)–(ix). *Then among all polynomials in K there is exactly one best approximation to f.*

Proof. Let P_0 be a minimal polynomial of best approximation for f. Assume that the exact degree of P_0 is ν_0. Assume that there is another polynomial P of best approximation. Then degree of $P \leqslant \nu_0$. Define $D = P_0 - P$. Let ν be the exact degree of D. Then $\nu \leqslant \nu_0$. Let E_{+}, E_{-}, E_{+}^{i}, E_{-}^{i} be those for P_0 and f, and m_{+}, m_{-}, l_{+}^{i}, l_{-}^{i} the numbers corre sponding to these sets. Since we assume $P_0 \neq P$ we see that m_{+} and m_{-} are finite, otherwise $D = P_0 - P \equiv 0$ and we would be done. Also since $\deg D = \nu$ we see that l_{+}^{i} and l_{-}^{i} are finite for all i for which $k_i \leqslant \nu$.

Let x_j, $j = 1,..., m_{+} + m_{-}$ represent the points of $E_{+} \cup E_{-}$ and let y_{ji}^{+}, y_{ji}^{-} represent the points of E_{+}^{i} and E_{-}^{i}, respectively. D satisfies the following conditions:

$$\begin{aligned}
D(x_i) &= 0 \qquad i = 1,..., m_{+} + m_{-}, \\
D^{(k_j)}(y_{ji}^{+}) &= 0 \qquad k_j \leqslant \nu,\ i = 1,..., l_{+}^{j}, \\
D^{(k_j)}(y_{ji}^{-}) &= 0 \qquad k_j \leqslant \nu,\ i = 1,..., l_{-}^{j}, \\
D^{(k_j+1)}(y_{ji}^{+}) &= 0 \qquad a < y_{ji}^{+} < b,\ k_j \leqslant \nu,\ i = 1,..., l_{+}^{j}, \\
D^{(k_j+1)}(y_{ji}^{-}) &= 0 \qquad a < y_{ji}^{-} < b,\ k_j \leqslant \nu,\ i = 1,..., l_{-}^{j}.
\end{aligned} \tag{14}$$

Let E be the incidence matrix corresponding to (14). The incidence matrix corresponding to these conditions is exactly the E of the previous lemma. Let N represent the total number of 1's in E. Then since E satisfies the Polya condition we have

$$N = \sum_{j=0}^{\nu} m_j \geqslant \nu + 1.$$

So, if necessary, we may extend E by adding columns of zeros numbering from $\nu + 1$ through $N - 1$. If $N = \nu + 1$ this is not necessary. This assures then, by the A–S theorem, that E is free. Thus $D \equiv 0$ since the only polynomial of degree $\leqslant N - 1$ satisfying (14) is identically zero and $\nu \leqslant N - 1$.

We note that in Lemma 2 if $k_1 \geqslant 1$ E satisfies the *strong Polya condition*

$$\sum_{j=0}^{k} m_j \geqslant k + 2, \qquad k = 0, 1,..., \nu - 1. \tag{**}$$

The proof is an obvious modification of the proof of Lemma 2.

The authors have made no attempt to obtain a Remez algorithm for this case as in [7], nor have they attempted to consider the case for $L_1[a, b]$ as in [3]. These remain open questions.

The authors have learned that Chalmers [8] has generalized these results in a more recent paper to appear in *Transactions of the American Mathematical Society*. In this paper this problem occurs as a special case.

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